The Solution of Certain Problems of Plane Strain in Laminated Orthotropic Structures by Means of Polynomials

By

ROBERT GEORGE BLAKE

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NOTE

Reference to an equation by a single number, thus (13) is reference to an equation of that number in the same section of the dissertation. Reference to equations in other sections of the dissertation is made by giving the section number followed by the number of the equation; for example, (2-5) refers to equation 5 in section 2.

CONTENTS

PLANE STRAIN IN LAMINATED ORTHOTROPIC STRUCTURES

0. Introduction. It is well known in the theory of plane strain in orthotropic materials that if F satisfies the equation

$$\gamma \frac{\partial^4 F}{\partial x^4} + 2(\nu - \beta)\frac{\partial^4 F}{\partial x^2 \partial y^2} + \alpha \frac{\partial^4 F}{\partial y^4} = 0 \qquad (1)$$

where α, β, γ, and ν depend upon the elastic constants of the material, then the stresses are given by

$$\tau_{xx} = \frac{\partial^2 F}{\partial y^2}, \quad \tau_{yy} = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}. \qquad (2)$$

The solution must also satisfy the boundary conditions. For laminated material it is also required that the stresses τ_{yy}, τ_{xy}, and the displacements, u, v, be continuous at the layers of separation. It is shown in this paper that polynomial solutions of (1) fail to meet this condition except for three rather trivial cases. Approximate methods based upon the concept of function space and others upon replacing the differential equation by finite equations are then developed. Several examples are worked out.

1. Plane strain. In problems of plane strain, with no body forces, the stress components τ_{xx}, τ_{yy}, and τ_{xy}, must satisfy the equations of equilibrium

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = 0 \qquad (1)$$

and

$$\tau_{zx} = \tau_{yz} = 0. \qquad (2)$$

The infinitesimal displacements are denoted by u and v (with w = 0). The strain components are defined by

$$e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad (3)$$

with $e_{xz} = e_{yz} = e_{zz} = 0$. The strain components must satisfy the compatibility equation

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y}. \qquad (4)$$

For orthotropic material where the planes of elastic symmetry are taken as the coordinate planes, Hooke's Law may be written

$$e_{xx} = \tau_{xx}/E_x - \sigma_{yx}\tau_{yy}/E_y - \sigma_{zx}\tau_{zz}/E_z$$

$$e_{yy} = -\sigma_{xy}\tau_{xx}/E_x + \tau_{yy}/E_y - \sigma_{zy}\tau_{zz}/E_z$$

$$e_{zz} = -\sigma_{xz}\tau_{xx}/E_x - \sigma_{yz}\tau_{yy}/E_y + \tau_{zz}/E_z$$

$$e_{xy} = \tau_{xy}/2\mu_{xy} \tag{5}$$

where E_x, E_y, and E_z, are the Young's moduli in the directions of the respective axes; σ_{ij} is Poisson's ratio, the ratio of the contraction parallel to the j-axis to the extension parallel to the i-axis associated with a tension parallel to the i-axis; and μ_{xy} is the modulus of rigidity associated with the x and y axes.

Since $e_{zz} = 0$, we have

$$\tau_{zz} = \sigma_{xz}\tau_{xx}E_z/E_x + \sigma_{yz}\tau_{yy}E_z/E_y. \tag{6}$$

Substituting in (5) we get

$$e_{xx} = \alpha\tau_{xx} - \beta\tau_{yy}$$

$$e_{yy} = -\beta\tau_{xx} + \gamma\tau_{yy}$$

$$e_{xy} = \nu\tau_{xy} \tag{7}$$

where

$$\alpha = (1 - \sigma_{xz}\sigma_{zx})/E_x$$

$$\beta = (\sigma_{xy} + \sigma_{xz}\sigma_{zy})/E_x = (\sigma_{yx} + \sigma_{yz}\sigma_{zx})/E_y$$

$$\gamma = (1 - \sigma_{yz}\sigma_{zy})/E_y$$

$$\nu = 1/(2\mu_{xy}) \tag{8}$$

and $$\sigma_{ij}/E_i = \sigma_{ji}/E_j. \tag{9}$$

Solving (7) for τ_{xx}, τ_{yy}, and τ_{xy} gives:

$$\begin{aligned} \tau_{xx} &= \omega e_{xx} + \phi e_{yy} \\ \tau_{yy} &= \phi e_{xx} + \lambda e_{yy} \\ \tau_{xy} &= 2\mu_{xy} e_{xy} \end{aligned} \tag{10}$$

where
$$\begin{aligned} \lambda &= \alpha/(\alpha\gamma - \beta^2) \\ \phi &= \beta/(\alpha\gamma - \beta^2) \\ \omega &= \gamma/(\alpha\gamma - \beta^2). \end{aligned} \tag{11}$$

If a stress function F can be found such that

$$\tau_{xx} = \frac{\partial^2 F}{\partial y^2}, \quad \tau_{yy} = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}. \tag{12}$$

then the equilibrium equations (1) are satisfied. Substituting in (7) gives

$$\begin{aligned} e_{xx} &= \alpha\frac{\partial^2 F}{\partial y^2} - \beta\frac{\partial^2 F}{\partial x^2} \\ e_{yy} &= -\beta\frac{\partial^2 F}{\partial y^2} + \gamma\frac{\partial^2 F}{\partial x^2} \\ e_{xy} &= -\nu\frac{\partial^2 F}{\partial x \partial y}. \end{aligned} \tag{13}$$

Substituting (13) in the compatibility equation (4) gives

$$\gamma\frac{\partial^4 F}{\partial x^4} + 2(\nu - \beta)\frac{\partial^4 F}{\partial x^2 \partial y^2} + \alpha\frac{\partial^4 F}{\partial y^4} = 0. \tag{14}$$

Let $\eta = (\gamma/\alpha)^{\frac{1}{4}}y$ and $K = (\nu - \beta)/(\alpha\gamma)^{\frac{1}{2}}$, then equation (14) becomes

$$\frac{\partial^4 F}{\partial x^4} + 2K\frac{\partial^4 F}{\partial x^2 \partial \eta^2} + \frac{\partial^4 F}{\partial \eta^4} = 0. \tag{15}$$

The solutions of (15) are of the form

$$F = R\{F_1(z_1) + F_2(z_2)\} \tag{16}$$

where
$$\begin{aligned} z_1 &= x + i\delta_1 y = x + i[K + (K^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}}\eta \\ z_2 &= x + i\delta_2 y = x + i[K - (K^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}}\eta \end{aligned} \tag{17}$$

with
$$\begin{aligned} \delta_1 &= (\gamma/\alpha)^{\frac{1}{4}}[K + (K^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}} \\ \delta_2 &= (\gamma/\alpha)^{\frac{1}{4}}[K - (K^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}}. \end{aligned} \tag{18}$$

In terms of F, the stress and strain components are given by

$$\begin{aligned} \tau_{xx} &= -\delta_1{}^2 R\left\{\frac{d^2F_1}{dz_1{}^2}\right\} - \delta_2{}^2 R\left\{\frac{d^2F_2}{dz_2{}^2}\right\} \\ \tau_{yy} &= R\left\{\frac{d^2F_1}{dz_1{}^2}\right\} + R\left\{\frac{d^2F_2}{dz_2{}^2}\right\} \\ \tau_{xy} &= \delta_1 R\left\{-i\frac{d^2F_1}{dz_1{}^2}\right\} + \delta_2 R\left\{-i\frac{d^2F_2}{dz_2{}^2}\right\}, \end{aligned} \tag{19}$$

and

$$e_{xx} = -\epsilon_1 R\left\{\frac{d^2F_1}{dz_1^2}\right\} - \epsilon_2 R\left\{\frac{d^2F_2}{dz_2^2}\right\}$$

$$e_{yy} = \zeta_1 R\left\{\frac{d^2F_1}{dz_1^2}\right\} + \zeta_2 R\left\{\frac{d^2F_2}{dz_2^2}\right\}$$

$$e_{xy} = \nu\delta_1 R\left\{-i\frac{d^2F_1}{dz_1^2}\right\} + \nu\delta_2 R\left\{-i\frac{d^2F_2}{dz_2^2}\right\}. \tag{20}$$

where

$$\epsilon_1 = \alpha\delta_1^2 + \beta, \quad \zeta_1 = \beta\delta_1^2 + \gamma. \tag{21}$$

Since

$$\delta_1^2\epsilon_1 + \zeta_1 - 2\nu\delta_1^2 = \alpha\delta_1^4 + 2\beta\delta_1^2 + \gamma - 2\nu\delta_1^2$$

$$= \alpha\delta_1^2 - 2(\nu - \beta)\delta_1^2 + \gamma = 0$$

we have

$$\zeta_1 = \delta_1^2(2\nu - \epsilon_1) \tag{22}$$

and also

$$\delta_1^2\delta_2^2 = \gamma/\alpha \tag{23}$$

$$\delta_1^2 + \delta_2^2 = 2(\nu - \beta)/\alpha. \tag{24}$$

2. Stresses, strains, and displacements. Let

$$F_1 = \sum_{n=0}^{m} (P_n + iM_n) z_1^{n+2}$$

$$F_2 = \sum_{n=0}^{m} (Q_n + iN_n) z_2^{n+2} . \tag{1}$$

Then

$$\frac{d^2F_1}{dz_1^2} = \sum_{\kappa=0}^{m} \sum_{p=0}^{m-\kappa} (\kappa+p+2)(\kappa+p+1)(P_{\kappa+p} + iM_{\kappa+p}) i^p \binom{\kappa+p}{\kappa} \delta_1^p x^\kappa y^p$$

$$\frac{d^2F_2}{dz_2^2} = \sum_{\kappa=0}^{m} \sum_{p=0}^{m-\kappa} (\kappa+p+2)(\kappa+p+1)(Q_{\kappa+p} + iN_{\kappa+p}) i^p \binom{\kappa+p}{\kappa} \delta_2^p x^\kappa y^p. \tag{2}$$

Substituting in (1-19) gives

$$\tau_{xx} = R\left\{\sum_{\kappa=0}^{m} \sum_{p=0}^{m-\kappa} (\kappa+p+2)(\kappa+p+1)(\delta_1^{p+2} P_{\kappa+p} + \delta_2^{p+2} Q_{\kappa+p} + i\delta_1^{p+2} M_{\kappa+p} + i\delta_2^{p+2} N_{\kappa+p}) i^{p+2} \binom{\kappa+p}{\kappa} x^\kappa y^p\right\}$$

$$= \sum_{\kappa=0}^{m} \Bigg[\sum_{\substack{p=0 \\ p \text{ even}}}^{\leq m-\kappa} (-1)^{\frac{1}{2}p+1} (\kappa+p+2)(\kappa+p+1) \binom{\kappa+p}{\kappa} (\delta_1^{p+2} P_{\kappa+p} + \delta_2^{p+2} Q_{\kappa+p}) y^p - \sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m-\kappa} (-1)^{\frac{1}{2}p+\frac{1}{2}} (\kappa+p+2)(\kappa+p+1) \binom{\kappa+p}{\kappa} \cdot (\delta_1^{p+2} M_{\kappa+p} + \delta_2^{p+2} N_{\kappa+p}) y^p \Bigg] x^\kappa \tag{3}$$

$$\tau_{yy} = \sum_{\kappa=0}^{m} \Bigg[\sum_{\substack{p=0 \\ p \text{ even}}}^{\leq m-\kappa} (-1)^{\frac{1}{2}p} (\kappa+p+2)(\kappa+p+1) \binom{\kappa+p}{\kappa} (\delta_1{}^p P_{\kappa+p}$$

$$+ \delta_2{}^p Q_{\kappa+p}) y^p + \sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m-\kappa} (-1)^{\frac{1}{2}p+\frac{1}{2}} (\kappa+p+2)(\kappa+p+1) \binom{\kappa+p}{\kappa}$$

$$\cdot (\delta_1{}^p M_{\kappa+p} + \delta_2{}^p N_{\kappa+p}) y^p \Bigg] x^\kappa. \tag{4}$$

$$\tau_{xy} = \sum_{\kappa=0}^{m} \Bigg[\sum_{\substack{p=0 \\ p \text{ even}}}^{\leq m-\kappa} (-1)^{\frac{1}{2}p} (\kappa+p+2)(\kappa+p+1) \binom{\kappa+p}{\kappa} (\delta_1{}^{p+1} M_{\kappa+p}$$

$$+ \delta_2{}^{p+1} N_{\kappa+p}) y^p + \sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m-\kappa} (-1)^{\frac{1}{2}p-\frac{1}{2}} (\kappa+p+2)(\kappa+p+1) \binom{\kappa+p}{\kappa}$$

$$\cdot (\delta_1{}^{p+1} P_{\kappa+p} + \delta_2{}^{p+1} Q_{\kappa+p}) y^p \Bigg] x^\kappa. \tag{5}$$

Substituting in (1-20) gives

$$e_{xx} = \sum_{\kappa=0}^{m} \Bigg[\sum_{\substack{p=0 \\ p \text{ even}}}^{\leq m-\kappa} (-1)^{\frac{1}{2}p+1} (\kappa+p+2)(\kappa+p+1) \binom{\kappa+p}{\kappa} (\delta_1{}^p \epsilon_1 P_{\kappa+p}$$

$$+ \delta_2{}^p \epsilon_2 Q_{\kappa+p}) y^p - \sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m-\kappa} (-1)^{\frac{1}{2}p+\frac{1}{2}} (\kappa+p+2)(\kappa+p+1) \binom{\kappa+p}{\kappa}$$

$$\cdot (\delta_1{}^p \epsilon_1 M_{\kappa+p} + \delta_2{}^p \epsilon_2 N_{\kappa+p}) y^p \Bigg] x^\kappa. \tag{6}$$

$$e_{yy} = \sum_{k=0}^{m} \Bigg[\sum_{\substack{p=0 \\ p \text{ even}}}^{\leq m-k} (-1)^{\frac{1}{2}p} (k+p+2)(k+p+1) \binom{k+p}{k} (\delta_1{}^p \zeta_1 P_{k+p}$$

$$+ \delta_2{}^p \zeta_2 Q_{k+p}) y^p + \sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m-k} (-1)^{\frac{1}{2}p+\frac{1}{2}} (k+p+2)(k+p+1) \binom{k+p}{k}$$

$$\cdot (\delta_1{}^p \zeta_1 M_{k+p} + \delta_2{}^p \zeta_2 N_{k+p}) y^p \Bigg] x^k, \tag{7}$$

$$e_{xy} = \nu \tau_{xy}. \tag{8}$$

Integrating (6) and (7) gives

$$u = \sum_{k=0}^{m} \Bigg[\sum_{\substack{p=0 \\ p \text{ even}}}^{\leq m-k} (-1)^{\frac{1}{2}p+1} (k+p+2) \binom{k+p+1}{k+1} (\delta_1{}^p \epsilon_1 P_{k+1} + \delta_2{}^p \epsilon_2 Q_{k+p}) y^p$$

$$- \sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m-k} (-1)^{\frac{1}{2}p+\frac{1}{2}} (k+p+2) \binom{k+p+1}{k+1} (\delta_1{}^p \epsilon_1 M_{k+p} + \delta_2{}^p \epsilon_2 N_{k+p}) y^p \Bigg]$$

$$\cdot x^{k+1} + f(y) \tag{9}$$

$$v = \sum_{k=0}^{m} \Bigg[\sum_{\substack{p=0 \\ p \text{ even}}}^{\leq m-k} (-1)^{\frac{1}{2}p} (k+p+2) \binom{k+p+1}{k} (\delta_1{}^p \zeta_1 P_{k+p} + \delta_2{}^p \zeta_2 Q_{k+p}) y^{p+1}$$

$$+ \sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m-k} (-1)^{\frac{1}{2}p+\frac{1}{2}} (k+p+2) \binom{k+p+1}{k} (\delta_1{}^p \zeta_1 M_{k+p} + \delta_2{}^p \zeta_2 N_{k+p}) y^{p+1} \Bigg]$$

$$\cdot x^k + g(x). \tag{10}$$

We determine $f(y)$ and $g(x)$ by substituting (9) and (10) in the third of the equations (1-3) and using (1-22) and (8). This gives

$$g(x) = \sum_{\kappa=0}^{m} (\kappa+2)(\delta_1{}^{-1}\zeta_1 M_\kappa + \delta_2{}^{-1}\zeta_2 N_\kappa)x^{\kappa+1} \tag{11}$$

$$\begin{aligned} f(y) = & \sum_{\substack{p=0 \\ p \text{ even}}}^{\leq m} (-1)^{\frac{1}{2}p}(p+2)(\delta_1{}^{p+1}\epsilon_1 M_p + \delta_2{}^{p+1}\epsilon_2 N_p)y^{p+1} \\ & + \sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m} (-1)^{\frac{1}{2}p-\frac{1}{2}}(p+2)(\delta_1{}^{p+1}\epsilon_1 P_p + \delta_2{}^{p+1}\epsilon_2 Q_p)y^{p+1}. \end{aligned} \tag{12}$$

3. Laminated construction. We assume that the material consists of three layers cemented together. The inner layer, or core, of thickness 2c will be distinguished by the subscript 1, and the two outer layers, or faces, which are assumed to be similar and of thickness b - c by the subscript 2. Thus E_{x1} is the Young's modulus of the core in the x-direction.

At the planes of separation, $y = \pm c$, it is necessary that τ_{yy}, τ_{xy}, u, and v be continuous. Equations expressing this condition are obtained from equations (2-4), (2-5), (2-9), and (2-10) by adding and then subtracting the value for y = c from that for y = -c and then setting the coefficients of each power of x equal to zero. The results are as follows:

$$\sum_{\substack{p=0 \\ p \text{ even}}}^{\leq m-k} (-1)^{\frac{1}{2}p}(k+p+2)(k+p+1)\binom{k+p}{k} c^p(\delta_{11}{}^p P_{k+p,1} + \delta_{21}{}^p Q_{k+p,1}$$

$$-\delta_{12}{}^p P_{k+p,2} - \delta_{22}{}^p Q_{k+p,2}) = 0 \qquad (1)$$

$$\sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m-k} (-1)^{\frac{1}{2}p-\frac{1}{2}}(k+p+2)(k+p+1)\binom{k+p}{k} c^p(\delta_{11}{}^{p+1} P_{k+p,1}$$

$$+\delta_{21}{}^{p+1} Q_{k+p,1} - \delta_{12}{}^{p+1} P_{k+p,2} - \delta_{22}{}^{p+1} Q_{k+p,2}) = 0 \qquad (2)$$

$$\sum_{\substack{p=0\\ p\ \text{even}}}^{\leq m-\kappa} (-1)^{\frac{1}{2}p}(\kappa+p+2)\binom{\kappa+p+1}{\kappa+1} c^p(\delta_{11}{}^p\epsilon_{11}P_{\kappa+p,1} + \delta_{21}{}^p\epsilon_{21}Q_{\kappa+p,1}$$

$$- \delta_{12}{}^p\epsilon_{12}P_{\kappa+p,2} - \delta_{22}{}^p\epsilon_{22}Q_{\kappa+p,2}) = 0 \tag{3}$$

$$\sum_{\substack{p=0\\ p\ \text{even}}}^{\leq m-\kappa} (-1)^{\frac{1}{2}p}(\kappa+p+2)\binom{\kappa+p+1}{\kappa} c^p(\delta_{11}{}^p s_{11}P_{\kappa+p,1} + \delta_{21}{}^p s_{21}Q_{\kappa+p,1}$$

$$- \delta_{12}{}^p s_{12}P_{\kappa+p,2} - \delta_{22}{}^p s_{22}Q_{\kappa+p,2}) = 0 \tag{4}$$

$$\sum_{\substack{p=1\\ p\ \text{odd}}}^{\leq m} (-1)^{\frac{1}{2}p-\frac{1}{2}}(p+2)c^p(\delta_{11}{}^{p+1}\epsilon_{11}P_{p,1} + \delta_{21}{}^{p+1}\epsilon_{21}Q_{p,1}$$

$$- \delta_{12}{}^{p+1}\epsilon_{12}P_{p,2} - \delta_{22}{}^{p+1}\epsilon_{22}Q_{p,2}) = 0 \tag{5}$$

$$\sum_{\substack{p=1\\ p\ \text{odd}}}^{\leq m-\kappa} (-1)^{\frac{1}{2}p+\frac{1}{2}}(\kappa+p+2)(\kappa+p+1)\binom{\kappa+p}{\kappa} c^p(\delta_{11}{}^p M_{\kappa+p,1} + \delta_{21}{}^p N_{\kappa+p,1}$$

$$- \delta_{12}{}^p M_{\kappa+p,2} - \delta_{22}{}^p N_{\kappa+p,2}) = 0 \tag{6}$$

$$\sum_{\substack{p=0\\ p\ \text{even}}}^{\leq m-\kappa} (-1)^{\frac{1}{2}p}(\kappa+p+2)(\kappa+p+1)\binom{\kappa+p}{\kappa} c^p(\delta_{11}{}^{p+1}M_{\kappa+p,1}$$

$$+ \delta_{21}{}^{p+1}N_{\kappa+p,1} - \delta_{12}{}^{p+1}M_{\kappa+p,2} - \delta_{22}{}^{p+1}N_{\kappa+p,2}) = 0 \tag{7}$$

$$\sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m-\kappa} (-1)^{\frac{1}{2}p+\frac{1}{2}}(\kappa+p+2)\binom{\kappa+p+1}{\kappa+1} c^p(\delta_{11}{}^p\epsilon_{11}M_{\kappa+p,1} + \delta_{21}{}^p\epsilon_{21}N_{\kappa+p,1}$$

$$- \delta_{12}{}^p\epsilon_{12}M_{\kappa+p,2} - \delta_{22}{}^p\epsilon_{22}N_{\kappa+p,2}) = 0 \qquad (8)$$

$$\sum_{\substack{p=-1 \\ p \text{ odd}}}^{\leq m-\kappa} (-1)^{\frac{1}{2}p+\frac{1}{2}}(\kappa+p+2)\binom{\kappa+p+1}{\kappa} c^{p+1}(\delta_{11}{}^p\zeta_{11}M_{\kappa+p,1}$$

$$+ \delta_{21}{}^p\zeta_{21}N_{\kappa+p,1} - \delta_{12}{}^p\zeta_{12}M_{\kappa+p,2} - \delta_{22}{}^p\zeta_{22}N_{\kappa+p,2}) = 0 \qquad (9)$$

$$\sum_{\substack{p=1 \\ p \text{ odd}}}^{\leq m} (-1)^{\frac{1}{2}p+\frac{1}{2}}(p+2)c^p(\delta_{11}{}^p\zeta_{11}M_{p,1} + \delta_{21}{}^p\zeta_{21}N_{p,1}$$

$$- \delta_{12}{}^p\zeta_{12}M_{p,2} - \delta_{22}{}^p\zeta_{22}N_{p,2}) = 0 \qquad (10)$$

In equations (1) - (4), (6), and (7) the range of κ is given by $0 \leq \kappa \leq m$, in equation (8) by $-1 \leq \kappa \leq m-1$, and in equation (9) by $1 \leq \kappa \leq m+1$.

If $m > 0$, then putting $\kappa = m$ in (1), (3), and (4), and $\kappa = m-1$ in (2), we get

$$P_{m1} + Q_{m1} - P_{m2} - Q_{m2} = 0$$

$$\delta_{11}{}^2P_{m1} + \delta_{21}{}^2Q_{m1} - \delta_{12}{}^2P_{m2} - \delta_{22}{}^2Q_{m2} = 0$$

$$\epsilon_{11}P_{m1} + \epsilon_{21}Q_{m1} - \epsilon_{12}P_{m2} - \epsilon_{22}Q_{m2} = 0$$

$$\zeta_{11}P_{m1} + \zeta_{21}Q_{m1} - \zeta_{12}P_{m2} - \zeta_{22}Q_{m2} = 0 \qquad (11)$$

In order for these to have a non-zero solution, it is necessary that

$$\begin{vmatrix} 1 & 1 & -1 & -1 \\ \delta_{11}^2 & \delta_{21}^2 & -\delta_{12}^2 & -\delta_{22}^2 \\ \epsilon_{11} & \epsilon_{21} & -\epsilon_{12} & -\epsilon_{22} \\ \zeta_{11} & \zeta_{21} & -\zeta_{12} & -\zeta_{22} \end{vmatrix}$$

$$= (\delta_{21}^2 - \delta_{11}^2)(\delta_{22}^2 - \delta_{12}^2)\left[(\beta_2-\beta_1)^2 - (\alpha_2-\alpha_1)(\gamma_2-\gamma_1)\right] = 0 \qquad (12)$$

Since

$$\delta_{21}^2 - \delta_{11}^2 = -2(\gamma_1/\alpha_1)^{\frac{1}{2}}(K_1^2 - 1)^{\frac{1}{2}}$$

the condition for non-zero roots reads:

At least one of the following relations must hold

$$K_1 = 1$$

$$K_2 = 1$$

$$(\beta_2 - \beta_1)^2 = (\alpha_2 - \alpha_1)(\gamma_2 - \gamma_1). \qquad (13)$$

For m = 0 equation (2) does not apply and we have the equations

$$P_{01} + Q_{01} - P_{02} - Q_{02} = 0$$

$$\epsilon_{11}P_{01} + \epsilon_{21}Q_{01} - \epsilon_{12}P_{02} - \epsilon_{22}Q_{02} = 0$$

$$\zeta_{11}P_{01} + \zeta_{21}Q_{01} - \zeta_{12}P_{02} - \zeta_{22}Q_{02} = 0 \qquad (14)$$

Their solution is

$$P_{01} = \frac{(\epsilon_{21}-\epsilon_{22})(\zeta_{22}-\zeta_{12}) - (\epsilon_{22}-\epsilon_{12})(\zeta_{21}-\zeta_{22})}{(\epsilon_{21}-\epsilon_{11})(\zeta_{11}-\zeta_{12}) - (\epsilon_{11}-\epsilon_{12})(\zeta_{21}-\zeta_{11})}Q_{02}$$

$$Q_{01} = \frac{(\epsilon_{22}-\epsilon_{11})(\zeta_{11}-\zeta_{12}) - (\epsilon_{11}-\epsilon_{12})(\zeta_{22}-\zeta_{11})}{(\epsilon_{21}-\epsilon_{11})(\zeta_{11}-\zeta_{12}) - (\epsilon_{11}-\epsilon_{12})(\zeta_{21}-\zeta_{11})}Q_{02}$$

$$P_{02} = -\frac{(\epsilon_{21}-\epsilon_{11})(\zeta_{11}-\zeta_{22}) - (\epsilon_{11}-\epsilon_{22})(\zeta_{21}-\zeta_{11})}{(\epsilon_{21}-\epsilon_{11})(\zeta_{11}-\zeta_{12}) - (\epsilon_{11}-\epsilon_{12})(\zeta_{21}-\zeta_{11})}Q_{02}. \qquad (15)$$

Putting $\kappa = m-1$ in (6) and (8), $\kappa = m$ in (7), and $\kappa = m+1$ in (9) gives, for $m > 0$,

$$\delta_{11}M_{m1} + \delta_{21}N_{m1} - \delta_{12}M_{m2} - \delta_{22}N_{m2} = 0$$

$$\delta_{11}\epsilon_{11}M_{m1} + \delta_{21}\epsilon_{21}N_{m1} - \delta_{12}\epsilon_{12}M_{m2} - \delta_{22}\epsilon_{22}N_{m2} = 0$$

$$\delta_{11}^{-1}\zeta_{11}M_{m1} + \delta_{21}^{-1}\zeta_{21}N_{m1} - \delta_{12}^{-1}\epsilon_{12}M_{m2} - \delta_{22}^{-1}\epsilon_{22}N_{m2} = 0. \qquad (16)$$

These have the solution

$$M_{m1} = -\frac{\delta_{22}(\epsilon_{22}-\epsilon_{12})}{\delta_{11}(\epsilon_{21}-\epsilon_{11})}N_{m2}, \quad N_{m1} = \frac{\delta_{22}(\epsilon_{22}-\epsilon_{12})}{\delta_{21}(\epsilon_{21}-\epsilon_{11})}N_{m2}, \quad M_{m2} = -\frac{\delta_{22}}{\delta_{12}}N_{m2}. \qquad (17)$$

Putting $\kappa = m-3$ in (6) and $\kappa = m-2$ in (7) gives, for $m > 2$

$$3!(\delta_{11}M_{m-2,1} + \delta_{21}N_{m-2,1} - \delta_{12}M_{m-2,2} - \delta_{22}N_{m-2,2})$$

$$= (m+2)(m+1)(\delta_{11}{}^3M_{m,1} + \delta_{21}{}^3N_{m,1} - \delta_{12}{}^3M_{m,2} - \delta_{22}{}^3N_{m,2})$$

$$2(\delta_{11}M_{m-2,1} + \delta_{21}N_{m-2,1} - \delta_{12}M_{m-2,2} - \delta_{22}N_{m-2,2})$$

$$= (m+2)(m+1)(\delta_{11}{}^3M_{m,1} + \delta_{21}{}^3N_{m,1} - \delta_{12}{}^3M_{m,2} - \delta_{22}{}^3N_{m,2}). \tag{18}$$

These are incompatible with equations (16). Therefore the only solutions to be considered are for $m = 0, 1, 2$.

For $m = 0$ we have only the equations (16) which have the solution (17).

For $m = 1$ we have equations (16) and in addition the following equation obtained from (10):

$$\delta_{11}s_{11}M_{11} + \delta_{21}s_{21}N_{11} - \delta_{12}s_{12}M_{12} - \delta_{22}s_{22}N_{12} = 0. \tag{19}$$

In order for these to have a non-zero solution, we must have

$$\begin{vmatrix} \delta_{11} & \delta_{21} & -\delta_{12} & -\delta_{22} \\ \delta_{11}\epsilon_{11} & \delta_{21}\epsilon_{21} & -\delta_{12}\epsilon_{12} & -\delta_{22}\epsilon_{22} \\ \delta_{11}^{-1}s_{11} & \delta_{21}^{-1}s_{21} & -\delta_{12}^{-1}s_{12} & -\delta_{22}^{-1}s_{22} \\ \delta_{11}s_{11} & \delta_{21}s_{21} & -\delta_{12}s_{12} & -\delta_{22}s_{22} \end{vmatrix} = 0. \tag{20}$$

In order for this condition to be satisfied, at least one of the following equalities must hold:

$$\nu_1 = \nu_2$$
$$K_1 = 1$$
$$K_2 = 1$$
$$\alpha_1\beta_2 = \alpha_2\beta_1. \tag{21}$$

For m = 2 we have the equations (16), the equations obtained for m = 1, and the following additional equations:

$$\delta_{11}M_{01} + \delta_{21}N_{01} - \delta_{12}M_{02} - \delta_{22}N_{02} - 6c^2(\delta_{11}{}^3M_{21} + \delta_{21}{}^3N_{21}$$
$$- \delta_{12}{}^3M_{22} - \delta_{22}{}^3N_{22}) = 0$$

$$\delta_{11}\epsilon_{11}M_{01} + \delta_{21}\epsilon_{21}N_{01} - \delta_{12}\epsilon_{12}M_{02} - \delta_{22}\epsilon_{22}N_{02} - 2c^2$$
$$\cdot(\delta_{11}{}^3\epsilon_{11}M_{21} + \delta_{21}{}^3\epsilon_{21}N_{21} - \delta_{12}{}^3\epsilon_{12}M_{22} - \delta_{22}{}^3\epsilon_{22}N_{22}) = 0$$

$$\delta_{11}^{-1}\zeta_{11}M_{01} + \delta_{21}^{-1}\zeta_{21}N_{01} - \delta_{12}^{-1}\zeta_{12}M_{02} - \delta_{22}^{-1}\zeta_{22}N_{02} - 6c^2$$
$$\cdot(\delta_{11}\zeta_{11}M_{21} + \delta_{21}\zeta_{21}N_{21} - \delta_{12}\zeta_{12}M_{22} - \delta_{22}\zeta_{22}N_{22}) = 0. \tag{22}$$

Unless (21) is satisfied, we must take $M_{11} = N_{11} = M_{12} = N_{12} = 0$. Substituting (17) in (22) gives

$$\delta_{11}M_{01}+\delta_{21}N_{01}-\delta_{12}M_{02} = \delta_{22}\left[N_{02}+6c^2(\epsilon_{22}-\epsilon_{12})(\alpha_1^{-1}-\alpha_2^{-1})N_{22}\right]$$

$$\delta_{11}\epsilon_{11}M_{01}+\delta_{21}\epsilon_{21}\cdot N_{01}-\delta_{12}\epsilon_{12}\cdot M_{02} = \delta_{22}\Big[\epsilon_{22}N_{02}$$
$$+2c^2(\epsilon_{22}-\epsilon_{12})(\delta_{11}^{\,2}+\delta_{21}^{\,2}-\delta_{12}^{\,2}-\delta_{22}^{\,2}+\beta_1\alpha_1^{-1}-\beta_2\alpha_2^{-1})N_{22}\Big]$$

$$\delta_{11}^{-1}S_{11}M_{01}+\delta_{21}^{-1}S_{21}N_{01}-\delta_{12}^{-1}S_{12}M_{02} = \delta_{22}\Big[\delta_{22}^{-2}S_{22}N_{02}$$
$$+6c^2(\epsilon_{22}-\epsilon_{12})(\beta_1\alpha_1^{-1}-\beta_2\alpha_2^{-1})N_{22}\Big]. \tag{23}$$

These have the solution

$$M_{01} = -\frac{\delta_{22}}{\delta_{11}}\left[\frac{K_2-\epsilon_{12}K_1}{\epsilon_{21}-\epsilon_{11}}+\frac{(\epsilon_{21}-\epsilon_{12})(K_3+K_2-2\nu_2K_1)}{2(\nu_2-\nu_1)(\epsilon_{21}-\epsilon_{11})}\right]$$

$$N_{01} = \frac{\delta_{22}}{\delta_{11}}\left[\frac{K_2-\epsilon_{12}K_1}{\epsilon_{21}-\epsilon_{11}}+\frac{(\epsilon_{11}-\epsilon_{12})(K_3+K_2-2\nu_2K_1)}{2(\nu_2-\nu_1)(\epsilon_{21}-\epsilon_{11})}\right]$$

$$M_{02} = -\frac{\delta_{22}}{\delta_{12}}\left[\frac{K_3+K_2-2\nu_1K_1}{2(\nu_2-\nu_1)}\right]. \tag{24}$$

where

$$K_1 = N_{02}+6c^2(\epsilon_{22}-\epsilon_{12})(\alpha_1^{-1}-\alpha_2^{-1})N_{22}$$

$$K_2 = \epsilon_{22}N_{02}$$
$$+2c^2(\epsilon_{22}-\epsilon_{12})(\delta_{11}^{\,2}+\delta_{21}^{\,2}-\delta_{12}^{\,2}-\delta_{22}^{\,2}+\beta_1\alpha_1^{-1}-\beta_2\alpha_2^{-1})N_{22}$$

$$K_3 = \delta_{22}^{-2}S_{22}N_{02}+6c^2(\epsilon_{22}-\epsilon_{12})(\beta_1\alpha_1^{-1}-\beta_2\alpha_2^{-1})N_{22}.$$

Thus we see that the only polynomials which give exact solutions are

$$F = P_0(x^2-\delta_1{}^2y^2) + Q_0(x^2-\delta_2{}^2y^2)$$

$$F = -\ 2M_0\delta_1 xy - 2N_0\delta_2 xy$$

$$F = -4M_2\delta_1 xy(x^2-\delta_1{}^2y^2)-2M_0\delta_1 xy-4N_2\delta_2 xy(x^2-\delta_2{}^2y^2)-2N_0\delta_2 xy. \tag{25}$$

4. Approximate solutions. Since so few polynomials give exact solutions, we must have recourse to approximation. One method of obtaining approximate solutions is by the use of function space.[1] We will give a brief introduction to the parts of this method which apply to our problem.

A vector $\vec{P}$ in the function space is the ordered set of functions $(\tau_{xx}, \tau_{yy}, \tau_{xy})$ which represent a state of stress of the body. The scalar product of two vectors is given by

$$\vec{P}\cdot\vec{P}' = \int(e_{11}\tau_{11}' + 2e_{12}\tau_{12}' + e_{22}\tau_{22}')dv$$
$$= \int(e_{11}'\tau_{11} + 2e_{12}'\tau_{12} + e_{22}'\tau_{22})dv \qquad (1)$$

where the e_{ij} are connected with the τ_{ij} by Hooke's law (1-5) and the integration is over the entire volume of the body. The length ρ of the vector $\vec{P}$ is given by

$$\rho^2 = |\vec{P}|^2 = \vec{P}\cdot\vec{P} \qquad (2)$$

The distance between two vectors is the length of the vector connecting them. Two vectors are said to be orthogonal if their scalar product vanishes.

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[1] W. Prager and J. L. Synge, "Approximation in elasticity based on the concept of function space." Quarterly of Applied Mathematics 5, 241-69 (1947).

An exact solution must satisfy:

(1) the equations of equilibrium,

(2) the equation of compatibility,

(3) the boundary conditions, and

(4) the continuity conditions at the layers of separation.

The stress state of such an exact solution is called the natural state and is denoted by $\vec{S}$.

If the boundary conditions are conditions on the stresses, or if such displacements as are specified are zero, the problem is said to be one with stress boundary conditions. In this case we have also the following notation:

$\vec{S}^*$ is called the completely associated state. It satisfies the equations of equilibrium, the boundary conditions on the stresses, and the condition of continuity on the stresses at the layers of separation.

$\vec{S}_p'$ where $p = 1, \ldots, m$ are called the homogeneous associated states. They satisfy the equations of equilibrium, the conditions of continuity on the stresses at the layers of separation, and have zero stresses at points on the boundary where the stresses are specified.

$\vec{S}_q''$ where $q = 1, \ldots, n$ are called complementary states. They satisfy the equation of compatibility, the conditions of continuity on displacements at the layers of separation, and give zero displacements at points on the boundary where zero displacements are specified.

$\vec{I}_p'$ where $p = 1, \ldots, m$ are called orthonormal homogeneous associated states. They are obtained from the set of homogeneous associated states by orthonormalizing.[1]

$\vec{I}_q''$ where $q = 1, \ldots, n$ are called orthonormal complementary states. They are obtained from the set of complementary states by orthonormalizing.

It is easily seen that:

I. The homogeneous associated states form a linear manifold $\mathfrak{M}$ called the homogeneous associated manifold.

II. If any homogeneous associated manifold be added to a completely associated state, the result is a completely associated state. The subspace of completely associated states $\mathfrak{M}^*$ is called the completely associated subspace.

III. The complementary states form a linear manifold $\mathfrak{N}$ called the complementary manifold.

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1 A convenient method of orthonormalizing is given in M. O. Peach, "Simplified technique for constructing orthonormal functions." Bul. Amer. Math. Soc. 50, 556-64 (1944).

The following theorems are proved in Prager-Synge:

IV. The homogeneous associated manifold and the complementary manifold are mutually orthogonal.

V. For the natural state $\vec{S}$ and any completely associated state $\vec{S}*$,

$$\vec{S}\cdot(\vec{S} - \vec{S}*) = 0. \tag{3}$$

Hence the extremity of $\vec{S}$ lies on a sphere having $\vec{S}*$ for a diameter; the center is at $\frac{1}{2}\vec{S}*$ and the radius is $\frac{1}{2}S*$.

VI. The natural state $\vec{S}$ is orthogonal to any homogeneous associated state $\vec{S}'$

$$\vec{S}\cdot\vec{S}' = 0. \tag{4}$$

Hence $\vec{S}$ is orthogonal to the homogeneous associated manifold m and

$$\vec{S}\cdot\vec{I}_p{}' = 0 \quad (p = 1, \ldots, m). \tag{5}$$

VII. The difference between the natural state $\vec{S}$ and any completely associated state $\vec{S}*$ is orthogonal to any complementary state $\vec{S}''$

$$(\vec{S} - \vec{S}*)\cdot\vec{S}'' = 0. \tag{6}$$

Hence $(\vec{S} - \vec{S}*)$ is orthogonal to the complementary manifold n and

$$\vec{S}\cdot\vec{I}_q{}'' = \vec{S}*\cdot\vec{I}_q{}'' \quad (q = 1, \ldots, n). \tag{7}$$

VIII. The difference between the natural state $\vec{S}$ and any completely associated state $\vec{S}*$ is orthogonal to the difference between $\vec{S}$ and any complementary state $\vec{S}''$:

$$(\vec{S} - \vec{S}*)\cdot(\vec{S} - \vec{S}'') = 0. \tag{8}$$

<u>Thus the natural state lies on every sphere which has for its diameter a line joining a point of the associated subspace $\mathfrak{M}^*$ to a point of the complementary manifold $\mathfrak{N}$.</u>

Obviously, the uncertainty in the position of $\vec{S}$ will be least if we pick a point of $\mathfrak{M}^*$ and of $\mathfrak{N}$ such that the distance between them is a minimum. To do that we let

$$\vec{M}^* = \vec{S}^* + \sum_{p=1}^{m} a_p \vec{I}_p'$$

be a point of $\mathfrak{M}^*$ and

$$\vec{N} = \sum_{q=1}^{n} b_q \vec{I}_q''$$

be a point of $\mathfrak{N}$. The square of the distance between them is

$$(\vec{M}^* - \vec{N})^2 = (\vec{S}^* + \sum_{p=1}^{m} a_p \vec{I}_p' - \sum_{q=1}^{n} b_q \vec{I}_q'')^2. \qquad (9)$$

A necessary condition for a minimum is

$$a_p = - \vec{S}^* \cdot \vec{I}_p', \qquad b_q = \vec{S}^* \cdot \vec{I}_q''.$$

Since $\mathfrak{M}^*$ and $\mathfrak{N}$ are mutually orthogonal, this is necessarily a minimum. We have then

$$\vec{V}_m^* = \vec{S}^* - \sum_{p=1}^{m} (\vec{S}^* \cdot \vec{I}_p') \vec{I}_p'$$

$$\vec{V}_n'' = \sum_{q=1}^{n} (\vec{S}^* . \vec{I}_q'') \vec{I}_q''. \qquad (10)$$

We define the error e of an approximation $\bar{S}$ by $\bar{S} - \vec{S}$. If any point on the sphere which has as its diameter $(\vec{V}_m* - \vec{V}_n")$ be chosen as $\bar{S}$, then we have

$$e^2 \leq (\vec{V}_m* - \vec{V}_n")^2. \tag{11}$$

If we choose for $\bar{S}$ the center of the sphere, which is given by

$$\vec{C} = \tfrac{1}{2}(\vec{V}_m* + \vec{V}_n") \tag{12}$$

the error satisfies the inequality

$$e^2 \leq \tfrac{1}{4}(\vec{V}_m* - \vec{V}_n")^2 = \tfrac{1}{4}(\vec{V}_m*^2 - \vec{V}_n"^2). \tag{13}$$

5. Application to laminated construction. As a completely associated state we can take

$$\vec{S}^* = (\tau_{xx}^*,\ \tau_{yy}^*,\ \tau_{xy}^*) = \left(\frac{\partial^2 F^*}{\partial y^2},\ \frac{\partial^2 F^*}{\partial x^2},\ -\frac{\partial^2 F^*}{\partial x \partial y}\right) \qquad (1)$$

where

$$F^* = R\{F_1^*(z_1) + F_2^*(z_2)\}$$

with

$$F_1^* = \sum_{n=0}^{m} (P_n^* + iM_n^*) z_1^{n+2}$$

$$F_2^* = \sum_{n=0}^{m} (Q_n^* + iN_n^*) z_2^{n+2}$$

and the P_n^*, Q_n^*, M_n^*, and N_n^* satisfying the equations (3-1), (3-2), (3-6), and (3-7).

Similarly for our first complementary state we can take

$$\vec{S}_1'' = (\tau_{1xx}'',\ \tau_{1yy}'',\ \tau_{1xy}'') = \left(\frac{\partial^2 F_1''}{\partial y^2},\ \frac{\partial^2 F_1''}{\partial x^2},\ -\frac{\partial^2 F_1''}{\partial x \partial y}\right) \qquad (2)$$

where F_1'' is defined in a manner analogous to F^* except that equations (3-3), (3-4), (3-5), (3-8), (3-9), and (3-10) must be satisfied.

Then

$$\vec{V}_0^* = \vec{S}^*$$

$$\vec{V}_1'' = (\vec{S}^* \cdot \vec{I}_1'') \cdot \vec{I}_1''. \qquad (3)$$

Since

$$\vec{I}_1'' = \vec{S}_1''/|\vec{S}_1''|$$

we have

$$\vec{V}_1'' = \frac{(\vec{S}^*\cdot\vec{S}_1'')\vec{S}_1''}{\vec{S}_1''\cdot\vec{S}_1''}. \qquad (4)$$

Using equations (1-19) and (1-20) we obtain

$$\begin{aligned}\vec{S}^*\cdot\vec{S}_1'' = \int\Big[&\delta_1{}^2\epsilon_1R_1{}^*R_1'' + \delta_1{}^2\epsilon_2R_1{}^*R_2'' + \delta_2{}^2\epsilon_1R_2{}^*R_1''\\ &+ \delta_2{}^2\epsilon_2R_2{}^*R_2'' + \zeta_1R_1{}^*R_1'' + \zeta_2R_1{}^*R_2'' + \zeta_1R_2{}^*R_1''\\ &+ \zeta_2R_2{}^*R_2'' + 2\nu(\delta_1{}^2I_1{}^*I_1'' + \delta_1\delta_2I_1{}^*I_2''\\ &+ \delta_1\delta_2I_2{}^*I_1'' + \delta_2{}^2I_2{}^*I_2'')\Big]\,dv \qquad (5)\end{aligned}$$

where

$R_1{}^*$ = the real part of $\dfrac{\partial^2F_1{}^*}{\partial z_1{}^2}$, ..., R_2'' = the real part of $\dfrac{\partial^2F_2''}{\partial z_2{}^2}$, ..., I_2'' = the imaginary part of $\dfrac{\partial^2F_2''}{\partial z_2{}^2}$.

Using (1-22), (1-24), and (1-11) equation (5) can be written

$$\begin{aligned}\vec{S}^*\cdot\vec{S}_1'' = \int 2\nu\Big[&\delta_1{}^2(R_1{}^*R_1'' + I_1{}^*I_1'') + \delta_2{}^2(R_2{}^*R_2'' + I_2{}^*I_2'')\\ &+ \left(\frac{\beta}{\alpha} + \frac{1}{\nu\lambda}\right)(R_1{}^*R_2'' + R_2{}^*R_1'') + \delta_1\delta_2(I_1{}^*I_2'' + I_2{}^*I_1'')\Big]dv.\end{aligned}$$

(6)

In a similar manner we obtain

$$\vec{S}* \cdot \vec{S}* = \int 2\nu\Big[\delta_1{}^2(R_1{}*^2 + I_1{}*^2) + \delta_2{}^2(R_2{}*^2 + I_2{}*^2)$$

$$+ 2\Big(\frac{\beta}{\alpha} + \frac{1}{\nu\lambda}\Big)R_1{}*R_2{}* + 2\delta_1\delta_2 I_1{}*I_2{}*\Big]dv \qquad (7)$$

and

$$\vec{S}_1{}'' \cdot \vec{S}_1{}'' = \int 2\nu\Big[\delta_1{}^2(R_1{}''^2 + I_1{}''^2) + \delta_2{}^2(R_2{}''^2 + I_2{}''^2)$$

$$+ 2\Big(\frac{\beta}{\alpha} + \frac{1}{\nu\lambda}\Big)R_1{}''R_2{}'' + 2\delta_1\delta_2 I_1{}''I_2{}''\Big]dv \qquad (8)$$

If we take

$$\overline{S} = \tfrac{1}{2}(\vec{V}_0{}* + \vec{V}_1{}''),$$

the maximum error can be found from

$$e^2 \leq \tfrac{1}{4}\Big[\vec{S}* \cdot \vec{S}* - \frac{(\vec{S}* \cdot \vec{S}_1{}'')^2}{\vec{S}_1{}'' \cdot \vec{S}_1{}''}\Big]. \qquad (9)$$

Any set of F_p' such that $\tau_{pxx}' = \dfrac{\partial^2 F_p'}{\partial y^2}$, $\tau_{pyy}' = \dfrac{\partial^2 F_p'}{\partial x^2}$, and $\tau_{pxy}' = -\dfrac{\partial^2 F_p'}{\partial x \partial y}$ equal zero on the boundary and such that τ_{pyy}' and τ_{pxy}' are continuous at $y = \pm c$ will serve to determine the $\vec{S}_p'$. Such a set is given by

$$F_p' = (x^2 - a^2)^2(y^2 - b^2)^2 f_p(x,y) \qquad (10)$$

where $f_p(x,y)$ is continuous and has continuous derivatives in the closed region occupied by the body.

A set of $\vec{S}''$ may be constructed from any set of displacements which take on zero values at points of the

boundary where the displacements are specified to be zero, and which are continuous at $y = \pm c$.

It is not necessary that the F^* used to determine the completely associated state $\vec{S}^*$ satisfy equation (1-15). Any F^* such that $\tau_{xx}^* = \frac{\partial^2 F^*}{\partial y^2}$, $\tau_{yy}^* = \frac{\partial^2 F^*}{\partial x^2}$, and $\tau_{xy}^* = -\frac{\partial^2 F^*}{\partial x \partial y}$ satisfy the boundary conditions and such that τ_{yy}^* and τ_{xy}^* are continuous at $y = \pm c$ can be used to obtain an approximate solution.

6. Illustrative examples. Let

$$F_1 = P_1 z_1^3 \qquad F_2 = Q_1 z_2^3. \tag{1}$$

Then

$$\vec{S}^* = [-6(\delta_1^2 P_1^* + \delta_2^2 Q_1^*)x,\ 6(P_1^* + Q_1^*)x,\ 6(\delta_1^2 P_1^* + \delta_2^2 Q_1^*)y] \tag{2}$$

where the values of P_1 and Q_1 in the first layer are related to their values in the second layer by the equations

$$P_{11}^* + Q_{11}^* - P_{12}^* - Q_{12}^* = 0$$

$$\delta_{11}^2 P_{11}^* + \delta_{21}^2 Q_{11}^* - \delta_{12}^2 P_{12}^* - \delta_{22}^2 Q_{12}^* = 0. \tag{3}$$

These have the solutions

$$P_{11}^* = [(\delta_{21}^2 - \delta_{12}^2)P_{12}^* + (\delta_{21}^2 - \delta_{22}^2)Q_{12}^*]/(\delta_{21}^2 - \delta_{11}^2)$$

$$Q_{11}^* = [(\delta_{12}^2 - \delta_{11}^2)P_{12}^* + (\delta_{22}^2 - \delta_{11}^2)Q_{12}^*]/(\delta_{21}^2 - \delta_{11}^2). \tag{4}$$

Also

$$\vec{S}_1'' = [-6(\delta_1^2 P_1'' + \delta_2^2 Q_1'')x,\ 6(P_1'' + Q_1'')x,\ 6(\delta_1^2 P_1'' + \delta_2^2 Q_1'')y] \tag{5}$$

with the equations

$$\epsilon_{11} P_{11}'' + \epsilon_{21} Q_{11}'' - \epsilon_{12} P_{12}'' - \epsilon_{22} Q_{12}'' = 0$$

$$\zeta_{11} P_{11}'' + \zeta_{21} Q_{11}'' - \zeta_{12} P_{12}'' - \zeta_{22} Q_{12}'' = 0, \tag{6}$$

which have the solutions

$$P_{11}'' = \frac{(\epsilon_{12}\zeta_{21}-\epsilon_{21}\zeta_{12})P_{12}'' + (\epsilon_{22}\zeta_{21}-\epsilon_{21}\zeta_{22})Q_{12}''}{\epsilon_{11}\zeta_{21} - \epsilon_{21}\zeta_{11}}$$

$$Q_{11}'' = \frac{(\epsilon_{11}\zeta_{12}-\epsilon_{12}\zeta_{11})P_{12}'' + (\epsilon_{11}\zeta_{22}-\epsilon_{22}\zeta_{11})Q_{12}''}{\epsilon_{11}\zeta_{21} - \epsilon_{21}\zeta_{11}}. \quad (7)$$

Let us further require that

$$P_{12}^* + Q_{12}^* = K$$

$$\delta_{12}^2 P_{12}^* + \delta_{22}^2 Q_{12}^* = 0 \quad (8)$$

and take

$$P_{12}'' = P_{12}^* \text{ and } Q_{12}'' = Q_{12}^* \quad (9)$$

These give

$$P_{12}'' = P_{12}^* = \delta_{22}^2 K/(\delta_{22}^2-\delta_{12}^2)$$

$$Q_{12}'' = Q_{12}^* = -\delta_{12}^2 K/(\delta_{22}^2-\delta_{12}^2)$$

$$P_{11}^* = \delta_{21}^2 K/(\delta_{21}^2-\delta_{11}^2)$$

$$Q_{11}^* = -\delta_{11}^2 K/(\delta_{21}^2-\delta_{11}^2). \quad (10)$$

$\tau_{xx1}^* = 0$ $\qquad \tau_{yy1}^* = 6Kx$ $\qquad \tau_{xy1}^* = 0$

$e_{xx1}^* = -6\beta_1 Kx$ $\qquad e_{yy1}^* = 6\gamma_1 Kx$ $\qquad e_{xy1}^* = 0$

$\tau_{xx2}^* = 0$ $\qquad \tau_{yy2}^* = 6Kx$ $\qquad \tau_{xy2}^* = 0$

$e_{xx2}^* = -6\beta_2 Kx$ $\qquad e_{yy2}^* = 6\gamma_2 Kx$ $\qquad e_{xy2}^* = 0$ $\qquad$ (11)

$$\tau_{1xx1}'' = \frac{6(\beta_1\gamma_2-\beta_2\gamma_1)Kx}{\alpha_1\gamma_1-\beta_1^2} \qquad \tau_{1yy1}'' = \frac{6(\alpha_1\gamma_2-\beta_1\beta_2)Kx}{\alpha_1\gamma_1-\beta_1^2} \qquad \tau_{1xy1}'' = \frac{6(\beta_2\gamma_1-\beta_1\gamma_2)Ky}{\alpha_1\gamma_1-\beta_1^2}$$

$$e_{1xx1}'' = -6\beta_2Kx \qquad e_{1yy1}'' = 6\gamma_2Kx \qquad e_{1xy1}'' = \nu_1\tau_{1xy1}''$$

$$\tau_{1xx2}'' = 0 \qquad \tau_{1yy2}'' = 6Kx \qquad \tau_{1xy2}'' = 0$$

$$e_{1xx2}'' = -6\beta_2Kx \qquad e_{1yy2}'' = 6\gamma_2Kx \qquad e_{1xy2}'' = 0 \tag{12}$$

$$\vec{S}*\cdot\vec{S}_1'' = 4\int_0^a x^2\left[\int_0^c 36\gamma_2K^2dy+\int_c^b 36\gamma_2K^2dy\right]dx = 48\gamma_2K^2a^3b \tag{13}$$

$$\vec{S}*\cdot\vec{S}* = 4\int_0^a x^2\left[\int_0^c 36\gamma_1K^2dy+\int_c^b 36\gamma_2K^2dy\right]dx = 48K^2a^3[c(\gamma_1-\gamma_2)+\gamma_2b] \tag{14}$$

$$\begin{aligned}\vec{S}_1''\cdot\vec{S}_1'' = {} & 4\int_0^a\Big\{\int_0^c\Big[36K^2x^2(\beta_2^2\gamma_1-2\beta_1\beta_2\gamma_2+\gamma_2^2\alpha_1)/(\alpha_1\gamma_1-\beta_1^2) \\ & + 72\nu_1K^2y^2(\beta_2\gamma_1-\beta_1\gamma_2)^2/(\alpha_1\gamma_1-\beta_1^2)^2\Big]dy+\int_c^b 36\gamma_2K^2x^2dy\Big\}dx \\ = {} & 48K^2\Big\{\left[(\beta_2^2\gamma_1-2\beta_1\beta_2\gamma_2-\alpha_1\gamma_2^2)/(\alpha_1\gamma_1-\beta_1^2)\right]a^3c \\ & + \left[2\nu_1(\beta_1\gamma_2-\beta_2\gamma_1)^2/(\alpha_1\gamma_1-\beta_1^2)^2\right]ac^3 + \gamma_2a^3(b-c)\Big\}.\end{aligned} \tag{15}$$

As a numerical example let us consider spruce with the longitudinal axis in the direction of the x-axis in the middle layer and the transverse axis in the direction of the x-axis in the outer layers, the radial axis being in the direction of the y-axis in both layers. Let us suppose that the layers are of equal thickness and that the width of the piece is ten times its thickness. We assume the

following to be the constants of the material:

$\alpha_1 = 0.507 \cdot 10^{-6}$ $\alpha_2 = 14.3 \cdot 10^{-6}$

$\beta_1 = 0.314 \cdot 10^{-6}$ $\beta_2 = 4.48 \cdot 10^{-6}$

$\gamma_1 = 6.48 \cdot 10^{-6}$ $\gamma_2 = 7.69 \cdot 10^{-6}$

$\nu_1 = 4.81 \cdot 10^{-6}$ $\nu_2 = 108.7 \cdot 10^{-6}$

We calculate the following results:

$\vec{S}^* \cdot \vec{S}_1'' = 29.9c^4K^2$, $\vec{S}^* \cdot \vec{S}^* = 28.3c^4K^2$, $\vec{S}_1'' \cdot \vec{S}_1'' = 78.6c^4K^2$

$\vec{V}_1'' = 0.380\vec{S}_1''$, $e \leq 2.05c^2K$, $e/\text{area} \leq 0.006K$.

We obtain, for our first approximation:

In the first layer

$\tau_{xx1} = -9.50Kx$ $\tau_{yy1} = 5.35Kx$ $\tau_{xy1} = 9.50Ky.$

In the second layer

$\tau_{xx2} = 0$ $\tau_{yy2} = 4.14Kx$ $\tau_{xy2} = 0.$

If we wish an approximation which satisfies the boundary values we take $\vec{V}_0^*$ as our solution. This gives, for both layers

$\tau_{xx} = 0$ $\tau_{yy} = 6Kx$ $\tau_{xy} = 0$

with an error of

$$e/\text{area} \leq 0.01K.$$

To improve this approximation we take

$$F_1' = (x^2-a^2)^2(y^2-b^2)^2x$$

This gives

$$\vec{S}_1{}' = \left[4x(x^2-a^2)^2(3y^2-b^2),\ 4x(5x^2-3a^2)(y^2-b^2)^2,\right.$$
$$\left. -4y(x^2-a^2)(5x^2-a^2)(y^2-b^2)\right]$$

$$\begin{aligned}\vec{S}_1{}'\cdot\vec{S}_1{}' = 256\,\big[&(32/3465)\alpha_3 a^{11}(9c^5/5 - 2b^2c^3 + b^4c)\\ &+ (16/315)\beta_3 a^9(3c^7/7 - 7b^2c^5/5 + 5b^4c^3/3 - b^6c)\\ &+ (1/7)\gamma_3 a^7(c^9/9 - 4b^2c^7/7 + 6b^4c^5/5 - 4b^6c^3/3 + b^8c)\\ &+ (64/315)\nu_3 a^9(c^7/7 - 2b^2c^5/5 + b^4c^3/3)\\ &+ (128/17325)\alpha_2 a^{11}b^5 - (512/33075)\beta_2 a^9 b^7\\ &+ (128/2205)\gamma_2 a^7 b^9 + (512/33075)\nu_2 a^9 b^7\big]\end{aligned}$$

where

$$\alpha_3 = \alpha_1-\alpha_2,\ \beta_3 = \beta_1-\beta_2,\ \gamma_3 = \gamma_1-\gamma_2,\ \nu_3 = \nu_1-\nu_2\ .$$

$$\vec{S}*\cdot\vec{S}_1{}' = -(256/35)\beta_3 a^7(c^3-b^2c)K$$

$$\vec{I}_1{}' = \vec{S}_1{}'/|\vec{S}_1{}'|$$

$$\vec{V}_1* = \vec{S}* - (\vec{S}*\cdot\vec{I}_1{}')\vec{I}_1{}' = \vec{S}* - (\vec{S}*\cdot\vec{S}_1{}'/\vec{S}_1{}'\cdot\vec{S}_1{}')\vec{S}_1{}'.$$

For the numerical case

$$\vec{S}_1{}'\cdot\vec{S}_1{}' = 9.93\cdot 10^{13}c^{16} \qquad \vec{S}*\cdot\vec{S}_1{}' = -5.34\cdot 10^6 c^{10}K$$

$$\vec{S}_1{}' = \left[12x(x^2-900c^2)^2(y^2-3c^2),\ 20x(x^2-540c^2)(y^2-9c^2)^2,\right.$$
$$\left. -20y(x^2-900c^2)(x^2-180c^2)(y^2-9c^2)\right]$$

$$\begin{aligned}\vec{V}_1* = \big[&4.69(x/30c)(x^2/900c^2 - 1)^2(y^2/3c^2 - 1)cK,\\ &180 + 1.41(x^2/540c^2 - 1)(y^2/9c^2 - 1)^2\ (x/30c)cK,\\ &-4.70(y/3c)(x^2/900c^2 - 1)(x^2/180c^2 - 1)(y^2/9c^2 - 1)cK\big]\end{aligned}$$

As a second example, let us consider the same problem except that the boundary conditions are

for $x = \pm 30c$ $(30c=a)$ $\tau_{xx} = K(1 - y^2/b^2)$, $\tau_{xy} = 0$

for $y = \pm 3c$ $(3c=b)$ $\tau_{yy} = 0$ $\tau_{xy} = 0.$

These boundary conditions will be satisfied if we take

$$F^* = \tfrac{1}{2}Ky^2(1 - y^2/6b^2).$$

If we also take

$$F_1' = (x^2-a^2)^2(y^2-b^2)^2$$

$$F_2' = x^2(x^2-a^2)^2(y^2-b^2)^2$$

$$F_3' = y^2(x^2-a^2)^2(y^2-b^2)^2$$

we obtain

$$\vec{S}^* = [K(1 - y^2/b^2),\ 0,\ 0]$$

$$\vec{S}_1' = [4(x^2-a^2)^2(3y^2-b^2),\ 4(3x^2-a^2)(y^2-b^2)^2,\ -16xy(x^2-a^2)(y^2-b^2)]$$

$$\vec{S}_2' = [4x^2(x^2-a^2)^2(3y^2-b^2),\ 2(15x^4-12a^2x^2+a^4)(y^2-b^2)^2,\ -8xy(3x^4-4a^2x^2+a^4)(y^2-b^2)]$$

$$\vec{S}_3' = [2(x^2-a^2)^2(15y^4-12b^2y^2+b^4),\ 4y^2(3x^2-a^2)(y^2-b^2)^2,\ -8xy(x^2-a^2)(3y^2-b^2)(y^2-b^2)]$$

$\vec{S}^*\cdot\vec{S}_1' = -2.86\cdot10^4c^8K$ $\quad$ $\vec{S}_1'\cdot\vec{S}_1' = 1.02\cdot10^{12}c^{14}$

$\vec{S}^*\cdot\vec{S}_2' = -3.68\cdot10^6c^{10}K$ $\quad$ $\vec{S}_1'\cdot\vec{S}_2' = 7.89\cdot10^{13}c^{16}$

$\vec{S}^*\cdot\vec{S}_3' = -6.59\cdot10^4c^{10}K$ $\quad$ $\vec{S}_1'\cdot\vec{S}_3' = -5.90\cdot10^{10}c^{16}$

$\vec{S}_2' \cdot \vec{S}_2' = 1.68 \cdot 10^{16} c^{18}$ $\vec{S}_3' \cdot \vec{S}_3' = 4.38 \cdot 10^{14} c^{18}$

$\vec{S}_2' \cdot \vec{S}_3' = -7.06 \cdot 10^{12} c^{18}$

Orthonormalizing gives

$$\vec{I}_1' = 9.90 \cdot 10^{-7} c^{-7} \vec{S}_1'$$

$$\vec{I}_2' = -7.48 \cdot 10^{-7} c^{-7} \vec{S}_1' + 9.67 \cdot 10^{-9} c^{-9} \vec{S}_2'$$

$$\vec{I}_3' = 1.90 \cdot 10^{-9} c^{-7} \vec{S}_1' + 1.12 \cdot 10^{-11} c^{-9} \vec{S}_2' + 4.77 \cdot 10^{-8} c^{-9} \vec{S}_3'$$

Then

$$\vec{V}_3* = \vec{S}* + 1.74 \cdot 10^{-8} c^{-6} K \vec{S}_1' + 1.37 \cdot 10^{-10} c^{-8} K \vec{S}_2' + 1.55 \cdot 10^{-10} c^{-8} K \vec{S}_3'.$$

We also take

$$\vec{S}_1'' = (\omega, \phi, 0)$$

then

$\vec{S}_1'' \cdot \vec{S}_1'' = 2.58 \cdot 10^8 c^2$ $\vec{S}* \cdot \vec{S}_1'' = 240 K c^2$

$\vec{I}_1'' = 6.23 \cdot 10^{-7} c^{-1} \vec{S}_1''$ $\vec{V}_1'' = 9.34 \cdot 10^{-7} K \vec{S}_1''$

$\tau_{1xx1}'' = 1.90K,$ $\tau_{1yy1}'' = 0.092K,$ $\tau_{1xy1}'' = 0$

$\tau_{1xx2}'' = 0.0553K,$ $\tau_{1yy2}'' = 0.0322K,$ $\tau_{1xy2}'' = 0.$

As a third example let us consider the same piece of spruce with the boundary conditions

for $x = 0$ $\tau_{xx} = 0,$ $\tau_{xy} = A(1 - y^2/b^2)$

$x = 60c$ $u = 0,$ $v = 0$

for $y = \pm b$ $(b=3c)$ $\tau_{yy} = 0$ $\tau_{xy} = 0$

These are satisfied if we take

$$F^* = -Axy(1 - y^2/3b^2)$$

giving

$$\vec{S}^* = \left[2Axy/b^2,\ 0,\ A(1 - y^2/b^2)\right].$$

We further take

$$F_1' = x^2y(y^2 - b^2)^2$$
$$F_2' = x^3y(y^2 - b^2)^2$$
$$F_3' = x^2y^3(y^2 - b^2)^2.$$

These give

$$\vec{S}_1' = \left[4x^2y(5y^2-3b^2),\ 2y(y^2-b^2)^2,\ -2x(5y^2-b^2)(y^2-b^2)\right]$$

$$\vec{S}_2' = \left[4x^3y(5y^2-3b^2),\ 6xy(y^2-b^2)^2,\ -3x^2(5y^2-b^2)(y^2-b^2)\right]$$

$$\vec{S}_3' = \left[2x^2y(21y^4-20b^2y^2+3b^4),\ 2y^3(y^2-b^2),\ -2xy^2(7y^2-3b^2)(y^2-b^2)\right]$$

$\vec{S}^*\cdot\vec{S}_1' = 7.30\cdot10^2c^7A$ $\qquad$ $\vec{S}_1'\cdot\vec{S}_1' = 7.62\cdot10^7c^{12}$

$\vec{S}^*\cdot\vec{S}_2' = 3.43\cdot10^4c^8A$ $\qquad$ $\vec{S}_1'\cdot\vec{S}_2' = 3.80\cdot10^9c^{13}$

$\vec{S}^*\cdot\vec{S}_3' = -4.28\cdot10^3c^9A$ $\qquad$ $\vec{S}_1'\cdot\vec{S}_3' = 3.61\cdot10^8c^{14}$

$\vec{S}_2'\cdot\vec{S}_2' = 1.95\cdot10^{11}c^{14}$ $\qquad$ $\vec{S}_3'\cdot\vec{S}_3' = 2.71\cdot10^9c^{16}$

$\vec{S}_2'\cdot\vec{S}_3' = 1.85\cdot10^{10}c^{15}$

$$\vec{I}_1' = 1.15\cdot10^{-4}c^{-6}\vec{S}_1'$$

$$\vec{I}_2' = -6.73\cdot10^{-4}c^{-6}\vec{S}_1' + 1.35\cdot10^{-5}c^{-7}\vec{S}_2'$$

$$\vec{I}_3' = -7.31\cdot10^{-6}c^{-6}\vec{S}_1' - 2.92\cdot10^{-6}c^{-7}\vec{S}_2' + 3.22\cdot10^{-5}c^{-8}\vec{S}_3'$$

$$\vec{V}_3* = \vec{S}* - 3.02\cdot10^{-5}c^{-5}\lambda\vec{S}_1' - 3.32\cdot10^{-7}c^{-6}\lambda\vec{S}_2' + 7.82\cdot10^{-6}c^{-7}\lambda\vec{S}_3'$$

The $\vec{S}_j''$ must satisfy the compatibility equation (1-4), give continuous displacements at $y=\pm c$, and satisfy $u=0$, $v=0$ at $x=60c$. One way to guarantee this is to start with a set of suitable displacements. Such a set is

$$u = (60c-x)f(x)g(y)$$
$$v = (60c-x)m(x)n(y)$$

where $f(x)$, $m(x)$, $g(y)$, $n(y)$ are bounded and continuous.

Let

$u_1'' = 60c-x$ $\qquad$ $v_1'' = 60c-x$

$u_2'' = (60c-x)y$ $\qquad$ $v_2'' = (60c-x)y$

These give

$e_{1xx}'' = -1$ $\qquad$ $e_{1yy}'' = 0$ $\qquad$ $e_{1xy}'' = -\frac{1}{2}$

$\tau_{1xx}'' = -\omega$ $\qquad$ $\tau_{1yy}'' = -\phi$ $\qquad$ $\tau_{1xy}'' = -\mu_{xy}$

$e_{2xx}'' = -y$ $\qquad$ $e_{2yy}'' = (60c-x)$ $\qquad$ $e_{2xy}'' = \frac{1}{2}(60c-x-y)$

$\tau_{2xx}'' = -\omega y+\phi(60c-x)$, $\qquad$ $\tau_{2yy}'' = -\phi y+\lambda(60c-x)$,

$\tau_{2xy} = \mu_{xy}(60c-x-y)$

$$\vec{S}_1'' = (-\omega,\ -\phi,\ -\mu_{xy})$$

$$\vec{S}_2'' = \left[-\omega y+\phi(60c-x),\ -\phi y+\lambda(60c-x),\ \mu_{xy}(60c-x-y)\right]$$

$\vec{S}*.\vec{S}_1'' = -2.40\cdot 10^2 Ac^2$ $\qquad$ $\vec{S}_1''\cdot\vec{S}_1'' = 2.71\cdot 10^8 c^2$

$\vec{S}*\cdot\vec{S}_2'' = -1.44\cdot 10^4 Ac^3$ $\qquad$ $\vec{S}_1''\cdot\vec{S}_2'' = -2.77\cdot 10^9 c^3$

$\vec{S}_2''\cdot\vec{S}_2'' = 7.12\cdot 10^{10} c^4$

$$\vec{I}_1'' = 6.08\cdot 10^{-4} c^{-1}\vec{S}_1''$$

$$\vec{I}_2'' = 4.93\cdot 10^{-5} c^{-1}\vec{S}_1'' + 4.82\cdot 10^{-6} c^{-2}\vec{S}_2''$$

$$\vec{V}_2'' = -9.64\cdot 10^{-5} A\vec{S}_1'' - 3.35\cdot 10^{-7} Ac^{-1}\vec{S}_2''.$$

For the solution we take the stress state given by

$$\vec{C} = \tfrac{1}{2}(\vec{V}_{3*} + \vec{V}_2'').$$

The values of the parameters in $\vec{S}_1''$ and $\vec{S}_2''$ are

In the middle layer	In the outer layers
$\omega = 2.03\cdot 10^6$	$\omega = 5.92\cdot 10^4$
$\phi = 9.85\cdot 10^4$	$\phi = 3.45\cdot 10^4$
$\lambda = 1.59\cdot 10^5$	$\lambda = 1.10\cdot 10^5$
$\mu_{xy} = 1.04\cdot 10^5$	$\mu_{xy} = 4.60\cdot 10^3$

7. Approximation by finite equations. Another method of approximation is to replace the differential equation by a set of finite equations. To do this we first approximate $F(x,y)$ by a polynomial $G(x,y)$ such that

$$G(x_i,y_j) = F(x_i,y_j) \qquad (1)$$

at certain points (x_i,y_j).

The given points are numbered in such a way that if the lines $x = x_i$ and $y = y_j$ for $i = 1, \ldots, I$ and $j = 1, \ldots, J$ are drawn in the xy-plane, the point (x_i,y_j) is the intersection of the lines $x = x_i$ and $y = y_j$. Then, from Lagrange's formula of interpolation, the required polynomial is

$$G(x,y) = \sum_{s_1}\sum_{r_1} \frac{\prod'_{s_2}(y-y_{s_2})}{\prod'_{s_2}(y_{s_1}-y_{s_2})} \frac{\prod'_{r_2}(x-x_{r_2})}{\prod'_{r_2}(x_{r_1}-x_{r_2})} F(x_r,y_s) \qquad (2)$$

where $\prod'_{r_j}$ indicates that in the product, r_j is not to take any of the values $r_j = r_1, r_2, \ldots, r_{j-1}$. If m is the only value in the range of r_2, we take

$$\frac{\prod'_{r_2}(x-x_{r_2})}{\prod'_{r_2}(x_{r_1}-x_{r_2})} = 1. \qquad (3)$$

Similar expressions involving y and s have similar meanings.

That $G(x_i, y_j) = F(x_i, y_j)$ can be seen from the fact that, if $x = x_m$ and $y = y_n$, where m belongs to the range of r and $1 \leqq n \leqq J$, then the following hold:

$$\frac{\prod'_{r_2}(x-x_{r_2})}{\prod'_{r_2}(x_{r_1}-x_{r_2})} = \begin{cases} 0 \text{ for } r_1 \neq m \\ 1 \text{ for } r_1 = m \end{cases}$$

$$\frac{\prod'_{s_2}(y-y_{s_2})}{\prod'_{s_2}(y_{s_1}-y_{s_2})} = \begin{cases} 0 \text{ for } s_1 \neq n \\ 1 \text{ for } s_1 = n \end{cases}$$

We replace the derivatives of F in the differential equations by the derivatives of G given by the following:

$$\frac{\partial G}{\partial x} = \sum_{s_1}\sum_{r_1} \frac{\prod'_{s_2}(y-y_{s_2}) \sum'_{r_2}\prod'_{r_3}(x-x_{r_3})}{\prod'_{s_2}(y_{s_1}-y_{s_2})\prod'_{r_2}(x_{r_1}-x_{r_2})} F(x_{r_1}, y_{s_1}) \tag{4}$$

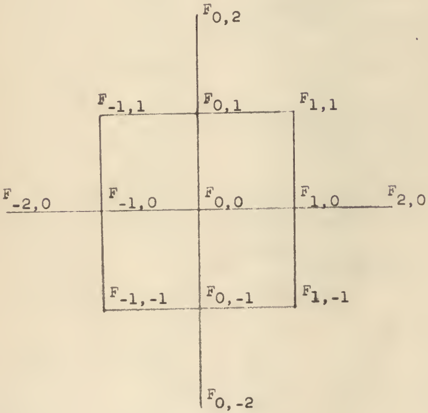
$$\frac{\partial^2 G}{\partial x^2} = \sum_{s_1}\sum_{r_1} \frac{\prod'_{s_2}(y-y_{s_2}) \sum'_{r_2}\sum'_{r_3}\prod'_{r_4}(x-x_{r_4})}{\prod'_{s_2}(y_{s_1}-y_{s_2})\prod'_{r_2}(x_{r_1}-x_{r_2})} F(x_{r_1}, y_{s_1}) \tag{5}$$

$$\frac{\partial G}{\partial y} = \sum_{s_1}\sum_{r_1} \frac{\sum'_{s_2}\prod'_{s_3}(y-y_{s_3})\prod'_{r_2}(x-x_{r_2})}{\prod'_{s_2}(y_{s_1}-y_{s_2})\prod'_{r_2}(x_{r_1}-x_{r_2})} F(x_{r_1}, y_{s_1}) \tag{6}$$

$$\frac{\partial^2 G}{\partial y^2} = \sum_{s_1}\sum_{r_1} \frac{\sum'_{s_2}\sum'_{s_3}\prod'_{s_4}(y-y_{s_4})\prod'_{r_2}(x-x_{r_2})}{\prod'_{s_2}(y_{s_1}-y_{s_2})\prod'_{r_2}(x_{r_1}-x_{r_2})} F(x_{r_1},y_{s_1}) \tag{7}$$

$$\frac{\partial^2 G}{\partial x\partial y} = \sum_{s_1}\sum_{r_1} \frac{\sum'_{s_2}\prod'_{s_3}(y-y_{s_3})\sum'_{r_2}\prod'_{r_3}(x-x_{r_3})}{\prod'_{s_2}(y_{s_1}-y_{s_2})\prod'_{r_2}(x_{r_1}-x_{r_2})} F(x_{r_1},y_{s_1}) \tag{8}$$

Let the lines $x=x_i$ and $y=y_j$ be evenly spaced at a distance of d apart, and let the values of $F_{ij} = F(x_i,y_j)$ be given at the points shown below.



(9)

Using (5), (7), and (8) we obtain

$$\left(\frac{\partial^2 e_{xx}}{\partial y^2}\right)_{0,0} = (1/12d^2)\left[-(e_{xx})_{0,-2}+16(e_{xx})_{0,-1}-30(e_{xx})_{0,0}\right.$$
$$\left.+16(e_{xx})_{0,1}-(e_{xx})_{0,2}\right] \tag{10}$$

$$\left(\frac{\partial^2 e_{yy}}{\partial x^2}\right)_{0,0} = (1/12d^2)\left[-(e_{yy})_{-2,0}+16(e_{yy})_{-1,0}-30(e_{yy})_{0,0}\right.$$
$$\left.+16(e_{yy})_{1,0}-(e_{yy})_{2,0}\right] \tag{11}$$

$$\left(\frac{\partial^2 e_{xy}}{\partial x \partial y}\right)_{0,0} = (1/3d^2)\left[(e_{xy})_{-1,-1}-(e_{xy})_{1,-1}-(e_{xy})_{-1,1}+(e_{xy})_{1,1}\right]. \tag{12}$$

Then using (1-13), (5), (7), and (8), and substituting the results in (1-4), we get the equation

$$\{-144\alpha_{0,-2}+96\alpha_{0,-1}+900\alpha_{0,0}+96\alpha_{0,1}-144\alpha_{0,2}-1800\beta_{0,0}-144\gamma_{-2,0}$$
$$+96\gamma_{-1,0}+900\gamma_{0,0}+96\gamma_{1,0}-144\gamma_{2,0}+216(\nu_{-1,-1}+\nu_{1,-1}+\nu_{-1,1}+\nu_{1,1})\}F_{0,0}$$
$$+(480\beta_{-1,0}+480\beta_{0,0}+104\gamma_{-2,0}-320\gamma_{-1,0}-480\gamma_{0,0}+64\gamma_{1,0}+56\gamma_{2,0}$$
$$-120\nu_{-1,-1}-72\nu_{1,-1}-120\nu_{-1,1}-72\nu_{1,1})F_{-1,0}+(104\alpha_{0,-2}-320\alpha_{0,-1}$$
$$-480\alpha_{0,0}+64\alpha_{0,1}+56\alpha_{0,2}+384\beta_{0,-1}-48\beta_{-2,0}+480\beta_{0,0}-48\beta_{2,0}$$
$$-160\nu_{-1,-1}-160\nu_{1,-1}-96\nu_{-1,1}-96\nu_{1,1})F_{0,-1}+(480\beta_{0,0}+480\beta_{1,0}$$
$$+56\gamma_{-2,0}+64\gamma_{-1,0}-480\gamma_{0,0}-320\gamma_{1,0}+104\gamma_{2,0}-72\nu_{-1,-1}-120\nu_{1,-1}$$
$$-72\nu_{-1,1}-120\nu_{1,1})F_{1,0}+(56\alpha_{0,-2}+64\alpha_{0,-1}-480\alpha_{0,0}-320\alpha_{0,1}$$
$$+104\alpha_{0,2}+384\beta_{0,1}-48\beta_{-2,0}+480\beta_{0,0}-48\beta_{2,0}-96\nu_{-1,1}-96\nu_{1,-1}$$
$$-160\nu_{-1,1}-160\nu_{1,1})F_{0,1}+(-192\beta_{0,-1}+48\beta_{-2,0}-256\beta_{-1,0}+16\beta_{2,0}$$

$$
\begin{aligned}
&+120\nu_{-1,-1}+40\nu_{1,-1}+72\nu_{-1,1}+24\nu_{1,1})F_{-1,-1}+(-192\beta_{0,-1}\\
&+16\beta_{-2,0}-256\beta_{1,0}+48\beta_{2,0}+40\nu_{-1,-1}+120\nu_{1,-1}+24\nu_{-1,1}+72\nu_{1,1})F_{1,-1}\\
&+(-192\beta_{0,1}+16\beta_{-2,0}-256\beta_{1,0}+48\beta_{2,0}+24\nu_{-1,-1}+72\nu_{1,-1}+40\nu_{1,1}\\
&+120\nu_{1,1})F_{1,1}+(-192\beta_{0,1}+48\beta_{-2,0}-256\beta_{-1,0}+16\beta_{2,0}+72\nu_{-1,-1}\\
&+24\nu_{1,-1}+120\nu_{-1,1}+40\nu_{1,1})F_{-1,1}+(-30\beta_{0,0}-30\beta_{-2,0}-35\gamma_{-2,0}\\
&+176\gamma_{-1,0}+30\gamma_{0,0}-16\gamma_{1,0}-11\gamma_{2,0}-36\nu_{-1,-1}+12\nu_{1,-1}-36\nu_{-1,1}\\
&+12\nu_{1,1})F_{-2,0}+(-35\alpha_{0,-2}+176\alpha_{0,-1}+30\alpha_{0,0}-16\alpha_{0,1}-11\alpha_{0,2}\\
&-\beta_{-2,0}+16\beta_{-1,0}-30\beta_{0,0}+16\beta_{1,0}-\beta_{2,0})F_{0,-2}+(-30\beta_{0,0}-30\beta_{2,0}\\
&-11\gamma_{-2,0}-16\gamma_{-1,0}+30\gamma_{0,0}+176\gamma_{1,0}-35\gamma_{2,0}+12\nu_{-1,-1}-36\nu_{1,-1}\\
&+12\nu_{-1,1}-36\nu_{1,1})F_{2,0}+(-11\alpha_{0,-2}-16\alpha_{0,-1}+30\alpha_{0,0}+176\alpha_{0,1}\\
&-35\alpha_{0,2}-\beta_{-2,0}+16\beta_{-1,0}-30\beta_{0,0}+16\beta_{1,0}-\beta_{0,2})F_{0,2}=0. \qquad (13)
\end{aligned}
$$

If α, β, γ, and ν are independent of the first subscripts, we have, where the second subscript is that given

$$
\begin{aligned}
&(-144\alpha_{-2}+96\alpha_{-1}+900\alpha_0+96\alpha_1-144\alpha_2-1800\beta_0+804\gamma_0+432\nu_{-1}+432\nu_1)F_{0,0}\\
&+(960\beta_0-576\gamma_0-192\nu_{-1}-192\nu_1)F_{-1,0}+(104\alpha_{-2}-320\alpha_{-1}-480\alpha_0\\
&+64\alpha_1+56\alpha_2+384\beta_{-1}+384\beta_0-320\nu_{-1}-192\nu_1)F_{0,-1}+(960\beta_0-576\gamma_0\\
&-192\nu_{-1}-192\nu_1)F_{1,0}+(56\alpha_{-2}+64\alpha_{-1}-480\alpha_0-320\alpha_1+104\alpha_2+384\beta_0\\
&384\beta_1-192\nu_{-1}-320\nu_1)F_{0,1}+(-384\beta_0+160\nu_{-1}+96\nu_1)F_{-1,-1}\\
&+(-384\beta_0+160\nu_{-1}+96\nu_1)F_{1,-1}+(-384\beta_0+96\nu_{-1}+160\nu_1)F_{1,1}\\
&+(-384\beta_0+96\nu_{-1}+160\nu_1)F_{-1,1}+(-60\beta_0+144\gamma_0-24\nu_{-1}-24\nu_1)F_{-2,0}\\
&+(-35\alpha_{-2}+176\alpha_{-1}+30\alpha_0-16\alpha_1-11\alpha_2)F_{0,-2}+(-60\beta_0+144\gamma_0-24\nu_{-1}\\
&-24\nu_1)F_{2,0}+(-11\alpha_{-2}-16\alpha_{-1}+30\alpha_0+176\alpha_1-35\alpha_2)F_{0,2}=0. \qquad (14)
\end{aligned}
$$

If all of the points lie in the same region, we have

$$(804\alpha-1800\beta+804\gamma+864\nu)F_{0,0} + (960\beta-576\gamma-384\nu)F_{-1,0} + (-576\alpha+768\beta-512\nu)F_{0,-1} + (960\beta-576\gamma-384\nu)F_{1,0} + (-576\alpha+768\beta-512\nu)F_{0,1} + (-384\beta+256\nu)F_{-1,-1} + (-384\beta+256\nu)F_{1,-1} + (-384\beta+256\nu)F_{1,1} + (-384\beta+256\nu)F_{-1,1} + (-60\beta+144\gamma-48\nu)F_{-2,0} + 144\alpha F_{0,-2} + (-60\beta+144\gamma-48\nu)F_{2,0} + 144\alpha F_{0,2} = 0 \qquad (15)$$

For points which lie on the boundary of two regions we take the average values for the values of α, β, γ, and ν.

If (0, 2) lies on the boundary and the rest of the points lie in region I, we have

$$(-72\alpha_{II}+876\alpha_{I}-1800\beta_{I}+804\gamma_{I}+864\nu_{I})F_{0,0} + (960\beta_{I}-576\gamma_{I}-384\nu_{I})F_{-1,0} + (28\alpha_{II}-604\alpha_{I}+768\beta_{I}+512\nu_{I})F_{0,-1} + (960\beta_{I}-576\gamma_{I}-384\nu_{I})F_{1,0} + (52\alpha_{II}-628\alpha_{I}+768\beta_{I}-512\nu_{I})F_{0,1} + (-384\beta_{I}+256\nu_{I})F_{-1,-1} + (-384\beta_{I}+256\nu_{I})F_{1,-1} + (-384\beta_{I}+256\nu_{I})F_{1,1} + (-384\beta_{I}+256\nu_{I})F_{-1,1} + (-60\beta_{I}+144\gamma_{I}-48\nu_{I})F_{-2,0} + (-5\tfrac{1}{2}\alpha_{II}+149\tfrac{1}{2}\alpha_{I})F_{0,-2} + (-60\beta_{I}+144\gamma_{I}-48\nu_{I})F_{2,0} + (-17\tfrac{1}{2}\alpha_{II}+161\tfrac{1}{2}\alpha_{I})F_{0,2} = 0 \qquad (16)$$

If the point (0, 2) lies in the region II, the points (-1, 1), (0, 1), and (1, 1) on the boundary, and the rest of the points in region I, we have

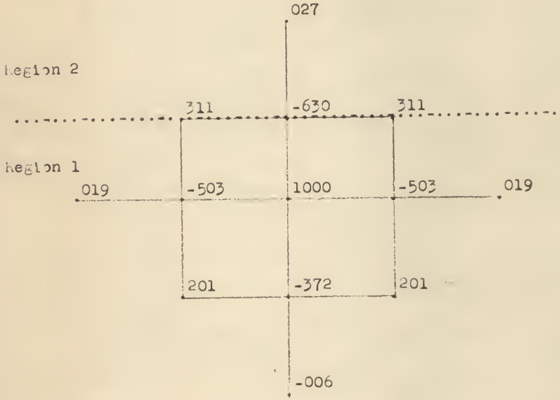
$$
\begin{aligned}
&(-96\alpha_{II}+900\alpha_{I}-1800\beta_{I}+804\gamma_{I}+216\nu_{II}+648\nu_{I})F_{0,0} + (960\beta_{I}-576\gamma_{I} \\
&-96\nu_{II}-288\nu_{I})F_{-1,0} + (88\alpha_{II}-664\alpha_{I}+768\beta_{I}-96\nu_{II}-416\nu_{I})F_{0,-1} \\
&+ (960\beta_{I}-576\gamma_{I}-96\nu_{II}-288\nu_{I})F_{1,0} + (-56\alpha_{II}-520\alpha_{I}+192\beta_{II} \\
&+576\beta_{I}-160\nu_{II}-352\nu_{I})F_{0,1} + (-384\beta_{I}+48\nu_{II}+208\nu_{I})F_{-1,-1} \\
&+ (-384\beta_{I}+48\nu_{II}+208\nu_{I})F_{1,-1} + (-384\beta_{I}+80\nu_{II}+176\nu_{I})F_{1,1} \\
&+ (-384\beta_{I}+80\nu_{II}+176\nu_{I})F_{-1,1} + (-60\beta_{I}+144\gamma_{I}-12\nu_{II}-36\nu_{I})F_{-2,0} \\
&+ (-19\alpha_{II}+163\alpha_{I})F_{0,-2} + (-60\beta_{I}+144\gamma_{I}-12\nu_{II}-36\nu_{I})F_{2,0} \\
&+ (53\alpha_{II}+91\alpha_{I})F_{0,2} = 0. \qquad (17)
\end{aligned}
$$

If the points (0, 2), (-1, 1), (0, 1), and (1, 1) lie in region II, the points (-2, 0), (-1, 0), (0, 0), (1, 0), and (2, 0) on the boundary, and the rest in region I, we have

$$
\begin{aligned}
&(402\alpha_{II}+402\alpha_{I}-900\beta_{II}-900\beta_{I}+402\gamma_{II}+402\gamma_{I}+432\nu_{II}+432\nu_{I})F_{0,0} \\
&+ (480\beta_{II}+480\beta_{I}-288\gamma_{II}+288\gamma_{I}-192\nu_{II}-192\nu_{I})F_{-1,0} + (-120\alpha_{II} \\
&-456\alpha_{I}+192\beta_{II}+576\beta_{I}-192\nu_{II}-320\nu_{I})F_{0,-1} + (480\beta_{II}+480\beta_{I} \\
&-288\gamma_{II}-288\gamma_{I}-192\nu_{II}-192\nu_{I})F_{1,0} + (-456\alpha_{II}-120\alpha_{I}+192\beta_{II} \\
&+576\beta_{I}-320\nu_{II}-192\nu_{I})F_{0,1} + (-192\beta_{II}-192\beta_{I}+92\nu_{II}+160\nu_{I})F_{-1,-1} \\
&+ (-192\beta_{II}-192\beta_{I}+96\nu_{II}+160\nu_{I})F_{1,-1} + (-192\beta_{II}-192\beta_{I}+160\nu_{II} \\
&+92\nu_{I})F_{1,1} + (-192\beta_{II}-192\beta_{I}+160\nu_{II}+96\nu_{I})F_{-1,1} + (-30\beta_{II} \\
&-30\beta_{I}+72\gamma_{II}+72\gamma_{I}-24\nu_{II}-24\nu_{I})F_{-2,0} + (-12\alpha_{II}+156\alpha_{I})F_{0,-2} \\
&+ (-30\beta_{II}-30\beta_{I}+72\gamma_{II}+72\gamma_{I}-24\nu_{II}-24\nu_{I})F_{2,0} + (156\alpha_{II} \\
&-12\alpha_{I})F_{0,2} = 0. \qquad (18)
\end{aligned}
$$

The other equations can be obtained by interchanging subscripts.

The above equations can be conveniently represented by diagrams similar to (9). The equations are then obtained by multiplying the value of F at each point by the number just to the right of the point on the diagram, adding the results, and equating the sum to zero. The following is an example of such a diagram for equation (17) using the constants given on page 33.



We calculate the value of F and its first derivatives by integrating the values of the stresses along the boundary. This gives us our boundary values for F. The

region is then divided into squares of length d by the lines $x=x_i$ and $y=y_j$. The intersections of these lines are called nodes. Nodes falling on the boundary are called boundary nodes. If some of the lines intersect the boundary in points other than nodes we call the node lying on the line, interior to the boundary, and closest to it, a boundary node. The value of F at such a boundary node is approximated from the values of F and its first and second derivatives at the boundary. The nodes adjacent to the boundary nodes and exterior to the boundary are called adjacent nodes. The values of F at adjacent nodes are approximated from the values of F and its first and second derivatives at the boundary. Nodes interior to the boundary which are not boundary nodes are called interior nodes. An equation can then be obtained for each interior node by use of the appropriate equation from (15) - (18). Thus we have replaced the differential equation by the set of finite equations:

$$\sum_{i,j} a_{ij}^{kl} F(x_i, y_j) = 0. \tag{19}$$

In general the number of equations given by (19) is too large to make a direct solution practical. The solution can, however, be approximated by the so-called relaxation method.

If we estimate the value of F as F^0 (taking the value of F as F^0 for a boundary node or an adjacent node) and substitute F^0 for F in the left side of (19) we get

$$\sum_{i,j} a_{ij}^{kl} F^0(x_i, y_j) = R^{kl} \tag{20}$$

If $R^{kl} = 0$ for all k, l, then $F^0 = F$ and the estimated solution is the correct solution. If this is not the case let

$$F'(x_k, y_l) = F^0(x_k, y_l) - R'^{kl}/a_{kl}^{kl}. \tag{21}$$

The kl'th equation then becomes

$$a_{kl}^{kl} F'(x_k, y_l) + \sum_{\substack{i \neq k \\ j \neq l}} a_{ij}^{kl} F^0(x_i, y_j) = R^{kl} - R'^{kl} = R''^{kl}.$$

The rest of the equations which involve $F(x_k, y_l)$ become

$$a_{kl}^{rs} F'(x_k, y_l) + \sum_{\substack{i \neq k \\ j \neq l}} a_{ij}^{rs} F^0(x_i, y_j) = R^{rs} - a_{kl}^{rs} R'^{kl}/a_{kl}^{kl}$$

Thus by changing the estimated value of $F(x_k, y_l)$ from $F^0(x_k, y_l)$ to $F'(x_k, y_l)$ the "residual" R^{kl} has been reduced to R''^{kl}. Starting with the largest $|R|$, each is relaxed in this manner until the largest remaining $|R|$ is negligible. The resulting F's give the approximate solution.

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BIOGRAPHICAL NOTE

Robert George Blake was born in Cornell, Illinois, on May 4, 1906. He is the son of Fred and Ethel Hunt Blake. He graduated from Cornell High School in 1924 and received a Junior College Diploma from Illinois State Normal University in 1926. He has received from the University of Florida the degrees of Bachelor of Arts in Education in 1938 and of Master of Arts in 1945. He is a member of Phi Kappa Phi. He has taught in the public schools of Livingston County, Illinois, and of Leon, Hernando, Alachua, and Lake Counties in Florida. He came to the University of Florida in 1943 as Instructor of Mathematics, War Training Program, and was promoted to the rank of Assistant Professor of Mathematics in 1949. In 1929 he married Genevieve Grelle of Brooksville, Florida. They have one daughter, Nancy.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council and was approved as a partial fulfilment of the requirements for the degree of Doctor of Philosophy.

January 31, 1953

C. F. Byers

Dean, College of Arts and Sciences

L. E. Grinter

Dean, Graduate School

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